

# A Note on Schnyder's Theorem

Fidel Barrera-Cruz\*

Penny Haxell†

Department of Combinatorics and Optimization  
University of Waterloo  
Waterloo, Ontario, Canada N2L 3G1

## Abstract

We give an alternate proof of Schnyder's Theorem, that the incidence poset of a graph  $G$  has dimension at most three if and only if  $G$  is planar.

**Keywords:** planar graphs, poset dimension, Schnyder's theorem

**AMS Subject Classification:** 05C10, 06A07

Order journal (July 2011) 28:221-226. Submitted: April 2010. Accepted: July 2010.

Final publication available at: <http://dx.doi.org/10.1007/s11083-010-9167-z>

## 1 Introduction

An important theorem of Schnyder [11] from 1989 relates two different notions of dimension for graphs  $G$ , namely the dimension of the incidence poset of  $G$ , and the minimum dimension in which  $G$  has a geometric realization. Schnyder's Theorem states that the incidence poset of  $G$  has dimension at most three if and only if  $G$  is planar. This work implies many useful results about planar graphs (see e.g. [11]) and motivated a significant body of further work, for example [2, 3, 4, 5, 6, 7, 8, 10].

Schnyder's proof developed a substantial amount of theory, involving notions of *normal labellings*, *dual orders*, and *tree decompositions* among others, which itself contributes much to the useful consequences of his theorem mentioned above. Each of the two implications of Schnyder's Theorem can also be derived from other works, for example [1, 10, 9]. However, we felt it could be useful to have a more direct proof, and the aim of this note is to provide one.

---

\*fbarreracruz@uwaterloo.ca; Partially supported by CONACYT, Project 209395/304106.

†pehaxell@math.uwaterloo.ca; Partially supported by NSERC.

## 2 Basics

Let  $V$  be a finite set of vertices. We will say that a set  $R = \{<_1, <_2, <_3\}$  of three linear orders of  $V$  is a *standard representation* of  $V$  if the following conditions hold.

- ( $R$  represents  $V$ ) The orders  $<_1, <_2, <_3$  have empty intersection, in other words, for every  $x \neq y$  in  $V$  there exists  $i \in \{1, 2, 3\}$  such that  $x <_i y$ .
- ( $R$  is *standard*) For each  $i$ , the maximum element  $a_i$  of  $<_i$  is among the two smallest elements of  $<_j$  for each  $j \neq i$ .

We write  $R = R(a_1, a_2, a_3)$  to indicate that  $a_i$  is the maximum element of  $<_i$  for each  $i$ . We define a graph  $\Sigma_2(R)$  with vertex set  $V$  as follows: the pair  $xy$  of distinct vertices is an edge of  $\Sigma_2(R)$  if and only if for every  $z \in V \setminus \{x, y\}$  there exists  $i \in \{1, 2, 3\}$  such that  $z >_i x$  and  $z >_i y$ . (The notation  $\Sigma_2(R)$  reflects the fact that this graph is part of the *complex*  $\Sigma(R)$  of  $R$ , see e.g. [10].) Then Schnyder's Theorem can be formulated as follows.

**Theorem 1** (*Schnyder's Theorem*). *Let  $G$  be a graph and suppose  $\{a_1, a_2, a_3\}$  is the vertex set of a triangle in  $G$ . Then  $G$  is a planar triangulation in which  $a_1 a_2 a_3$  is a face if and only if there exists a standard representation  $R = R(a_1, a_2, a_3)$  of  $V = V(G)$  such that  $G = \Sigma_2(R)$ .*

We begin by establishing some basic facts about the neighborhood (denoted by  $\Gamma$ ) of  $a_1$ , and the vertex  $b = \max_{<_1} V \setminus \{a_1\}$  in the graph  $\Sigma_2(R)$ .

**Lemma 2** *Let  $R = R(a_1, a_2, a_3)$  be a standard representation of  $V$ , and let the neighbors of  $a_1$  in  $\Sigma_2(R)$  be  $w_0 <_2 w_1 <_2 \dots <_2 w_m <_2 w_{m+1}$ . Then*

1.  $w_{m+1} <_3 w_m <_3 \dots <_3 w_1 <_3 w_0$ ,
2.  $w_0 = a_3$  and  $w_{m+1} = a_2$ ,
3. the set  $S_i = \{z \in V : w_i <_2 z <_2 w_{i+1} \text{ and } w_{i+1} <_3 z <_3 w_i\}$  is empty,
4.  $w_i w_{i+1} \in \Sigma_2(R)$  for each  $i$ ,
5. the vertex  $b$  is a neighbor  $w_i$  of  $a_1$  for some  $i$ . Moreover, if  $|V| \geq 4$  then  $a_1$  and  $b = w_i$  have exactly two common neighbors, namely  $w_{i-1}$  and  $w_{i+1}$ ,
6. if  $|V| \geq 4$  then every  $z \in \Gamma(b) \setminus \{a_1, w_{i-1}, w_{i+1}\}$  satisfies  $w_i <_2 z <_2 w_{i+1}$  and  $w_i <_3 z <_3 w_{i-1}$ .

**Proof.** For (1), suppose  $j < i$ . Since  $a_1 w_i \in \Sigma_2(R)$  we know there is an order  $k$  in which  $a_1, w_i <_k w_j$ . But  $k \neq 1$  since  $a_1$  is maximum in  $<_1$ , and  $k \neq 2$  by assumption. Thus  $k = 3$  as required.

For (2), since  $R$  is standard we know each  $z \in V \setminus \{a_1, a_2\}$  satisfies  $a_1, a_2 <_3 z$ , so  $a_1 a_2 \in \Sigma_2(R)$  and thus  $w_{m+1} = a_2$ . Similarly  $a_1 a_3 \in \Sigma_2(R)$ , and so  $w_0 = a_3$  since by (1) we know  $w_0$  is the neighbor of  $a_1$  that is highest in  $<_3$ .

For (3), suppose on the contrary that for some  $i$  we have  $y_\star = \min_{<_2} S_i$ . Then since  $y_\star a_1$  is not an edge of  $\Sigma_2(R)$ , and  $R$  is standard, there exists a vertex  $z \notin \{y_\star, a_1\}$  such that  $z <_2 y_\star$  and  $z <_3 y_\star$ . Since  $w_i a_1 \in \Sigma_2(R)$  we must have  $w_i <_2 z$ . Similarly, we have  $w_{i+1} <_3 z$ . This contradicts the minimality of  $y_\star$ , since  $w_{i+1} <_3 z <_3 y_\star <_3 w_i$  and  $w_i <_2 z <_2 y_\star <_2 w_{i+1}$ . This shows  $S_i = \emptyset$ .

For (4), suppose on the contrary that there exists  $y \in V \setminus \{w_i, w_{i+1}\}$  so that for each  $k$ , either  $y <_k w_i$  or  $y <_k w_{i+1}$ . By (1) this implies in particular that  $y <_2 w_{i+1}$  and  $y <_3 w_i$ . But since  $a_1 w_i \in \Sigma_2(R)$  and  $a_1 w_{i+1} \in \Sigma_2(R)$ , we must have  $w_i <_2 y$  and  $w_{i+1} <_3 y$ , implying  $y \in S_i$ . This contradicts (3).

For (5), let  $x \in V(G) \setminus \{a_1, b\}$ . Since  $R$  represents  $V$  we know that  $b <_j x$  for some  $j$ , and by definition of  $b$  we have  $j \neq 1$ . Thus, since  $R$  is standard, we conclude  $a_1 <_j b$  also, and hence  $a_1 b \in \Sigma_2(R)$ , so  $b = w_i$  for some  $i$ . Provided  $|V| \geq 4$ , since  $R$  is standard we know  $b \notin \{a_2, a_3\}$ , and so  $0 < i < m + 1$  by (2). Therefore by (4) we know  $w_{i-1}$  and  $w_{i+1}$  are distinct common neighbors of  $a_1$  and  $b$ . Now suppose on the contrary that  $w_j \notin \{w_{i-1}, w_{i+1}\}$  is a third common neighbor of  $a_1$  and  $b = w_i$ . Suppose  $j < i - 1$ . Then  $w_{i-1} <_3 w_j$ , so since  $w_{i-1} <_2 w_i$ , and  $w_{i-1} <_1 w_i$  by definition of  $b = w_i$ , we see that  $w_{i-1}$  is not above both of  $w_i$  and  $w_j$  in any order. This contradicts the fact that  $w_i w_j \in \Sigma_2(R)$ . Similarly we find a contradiction if  $j > i + 1$ .

For (6), let  $z$  be a neighbor of  $b = w_i$  different from  $a_1$ , and suppose on the contrary that  $z <_2 w_i$ . Again since  $|V| \geq 4$  we have  $0 < i < m + 1$ . By (4) we know  $w_i w_{i-1} \in \Sigma_2(R)$ , so we must have  $w_i <_3 w_{i-1} <_3 z$ . This implies there is no order in which  $w_{i-1}$  is above  $z$  and  $w_i$ , contradicting the fact that  $w_i z \in \Sigma_2(R)$ . Furthermore, we cannot have  $w_{i+1} <_2 z$ , since in that case  $w_{i+1}$  would not be above both  $w_i$  and  $z$  in any order. Similarly, we can show that  $w_i <_3 z <_3 w_{i-1}$  as required.  $\square$

### 3 Represented graphs are triangulations

In this section we prove that if  $R = R(a_1, a_2, a_3)$  is a standard representation of  $V$  then  $\Sigma_2(R)$  is a planar triangulation containing  $a_1 a_2 a_3$  as a face. Our approach will use induction on  $|V|$ . Suppose  $|V| \geq 4$ . For  $b = \max_{<_1} V \setminus \{a_1\}$ , we define a set  $R'$  of three linear orders of the set  $V' = V \setminus \{b\}$  by suppressing  $b$ , in other words we let  $<'_i = <_i|_{V \setminus \{b\}}$  and set  $R' = \{<'_1, <'_2, <'_3\}$ . Then it is clear from the definitions that  $R'$  is a standard representation of  $V'$ .

**Lemma 3** *With the above definitions, the graph  $\Sigma_2(R')$  is the graph  $H$  obtained from  $\Sigma_2(R)$  by contracting the edge  $a_1b$  and labelling the new vertex  $a_1$ .*

**Proof.** First we show that every edge of  $H$  is an edge of  $\Sigma_2(R')$ . Fix  $st \in H$ , and suppose  $z \in V' \setminus \{s, t\}$ . If  $st \in \Sigma_2(R)$ , then  $s, t <_j z$  for some  $j \in \{1, 2, 3\}$ . Thus  $s, t <'_j z$ , implying  $st \in \Sigma_2(R')$  as required. Otherwise  $st \notin \Sigma_2(R)$ , in which case  $a_1 \in \{s, t\}$ , say  $s = a_1$ , and  $t$  is a neighbor of  $b$  in  $\Sigma_2(R)$ . Thus since  $bt \in \Sigma_2(R)$ , there exists  $j \in \{1, 2, 3\}$  so that  $b, t <_j z$ . Since  $z \neq a_1$  we know  $j \neq 1$ , as there is no element  $z \neq a_1$  which satisfies  $b <_1 z$ . Since the representation is standard,  $a_1 <_j b$  if  $j \neq 1$ . So we have that  $a_1, t <'_j z$ , showing  $st \in \Sigma_2(R')$ .

Now we show that every edge  $st \in \Sigma_2(R')$  is an edge of  $H$ . If  $st \in \Sigma_2(R)$  then  $s, t \neq b$  so  $st \in H$ . If  $st \notin \Sigma_2(R)$  then some element of  $V$  is not above  $s$  and  $t$  in any order. Since  $st \in \Sigma_2(R')$  the only possibility for this element is  $b$ . This implies  $a_1 \in \{s, t\}$ , say  $s = a_1$ , and that  $b <_2 t$  and  $b <_3 t$ . By definition of contraction, it suffices to prove that  $t$  is a neighbor of  $b$  in  $\Sigma_2(R)$ . Let  $y \in V \setminus \{b, t\}$ . If  $y = a_1$ , then  $b, t <_1 y$ . If  $y \neq a_1$ , then since  $a_1t \in \Sigma_2(R')$  we know that  $a_1, t <'_2 y$  or  $a_1, t <'_3 y$ . This implies  $a_1, t <_2 y$  or  $a_1, t <_3 y$ . Therefore  $b <_2 t <_2 y$  or  $b <_3 t <_3 y$ . This shows  $bt \in \Sigma_2(R)$ , and the proof is complete.  $\square$

We are now ready to prove the main result of this section.

**Theorem 4** *Let  $V$  be a set with  $|V| \geq 3$  and let  $R = R(a_1, a_2, a_3)$  be a standard representation of  $V$ . Then  $\Sigma_2(R)$  is a planar triangulation containing  $a_1a_2a_3$  as a face.*

**Proof.** We will proceed by induction on  $|V|$ . The statement holds for  $|V| = 3$  and 4, so assume  $|V| \geq 5$  and that the statement is true for sets of size less than  $|V|$ . As usual we denote by  $b$  the second-largest element of  $<_1$ , and by  $w_0, \dots, w_{m+1}$  the neighbors of  $a_1$  as in Lemma 2. Then by Lemma 2(5)  $b = w_i$  for some  $0 < i < m+1$ .

Let  $R'$  be the representation of  $V' = V \setminus \{b\}$  obtained by suppressing  $b$ . Let  $z_0, z_1, \dots, z_{d+1}$  be the neighbors of  $b$  in  $\Sigma_2(R)$  that are different from  $a_1$ . (Note that  $d = 0$  is possible.) By Lemma 2(5) we may assume without loss of generality that  $z_0 = w_{i-1}$  and  $z_{d+1} = w_{i+1}$ , and that  $\{z_1, \dots, z_d\} \cap \{w_0, \dots, w_{m+1}\} = \emptyset$ . By (6) we may assume that  $w_i <_2 z_1 <_2 \dots <_2 z_d <_2 z_{d+1} = w_{i+1}$ . Then by Lemma 3 the neighborhood of  $a_1$  in  $\Sigma_2(R')$  is  $w_0 <_2 \dots <_2 w_{i-1} <_2 z_1 <_2 \dots <_2 z_d <_2 w_{i+1} <_2 \dots <_2 w_{m+1}$ . By Lemma 2(4) applied to  $R'$  we know that consecutive elements in this list are joined by an edge in  $\Sigma(R')$ , so since  $w_0w_{m+1} = a_3a_2$  is an edge of  $\Sigma_2(R')$ , this forms a cycle in  $\Sigma(R')$  whose vertex set is  $\Gamma(a_1)$ .

By the induction hypothesis  $\Sigma_2(R')$  is a planar triangulation with  $a_1a_2a_3$  as a face, so for convenience let us fix a planar drawing for which  $a_1a_2a_3$  is the outer face. Since in a planar triangulation there is a unique Hamilton cycle in the subgraph induced by  $\Gamma(v)$  for any vertex  $v$  (namely the cycle whose edges are  $\{xy :$

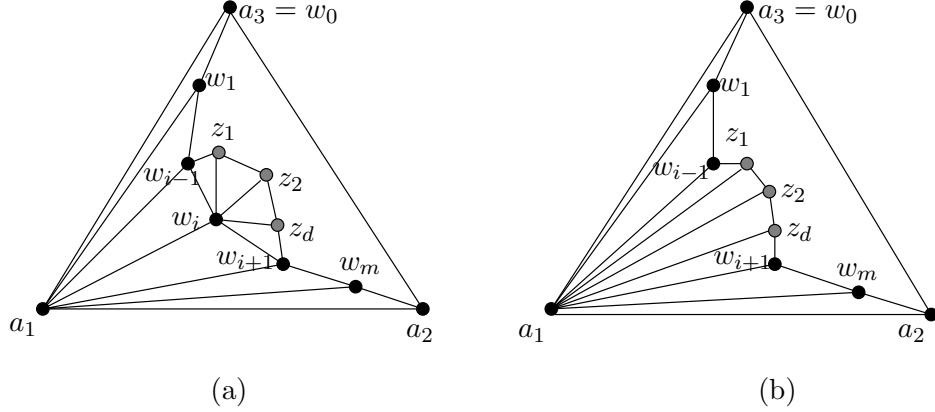


Figure 1: The neighborhood of  $a_1$

$xyv$  is a face}), we have a drawing as shown in Figure 1(b). We obtain a new planar drawing from this drawing of  $\Sigma_2(R')$  by removing all edges  $a_1z_j$  for  $1 \leq j \leq d$ , adding a new vertex  $w_i = b$  inside the region bounded by the cycle  $a_1z_0 \dots z_{d+1}$ , and joining  $w_i$  to all of  $a_1, z_0, \dots, z_{d+1}$  (see Figure 1(a)). This is a planar drawing of  $\Sigma_2(R)$ , because by Lemma 3 every edge of  $\Sigma_2(R)$  that is not an edge of  $\Sigma_2(R')$  is incident to  $b$ , and the set of edges of  $\Sigma_2(R)$  incident to  $b$  is precisely  $\{ba_1, bz_0, \dots, bz_{d+1}\}$ . Finally, we note that since  $\Sigma_2(R')$  is a triangulation and  $\Sigma_2(R)$  is a planar graph with one more vertex and three more edges, we see  $\Sigma_2(R)$  is a planar triangulation, and it has  $a_1a_2a_3$  as a face. Thus by induction the proof is complete.  $\square$

## 4 Triangulations are represented graphs

Our aim in this section is to prove the other implication of Theorem 1, as follows.

**Theorem 5** *Let  $G$  be a planar triangulation and let  $a_1a_2a_3$  be a face in  $G$ . Then there exists a standard representation  $R = R(a_1, a_2, a_3)$  of  $V = V(G)$  such that  $G = \Sigma_2(R)$ .*

**Proof.** Again we will use induction on  $|V|$ . If  $|V| = 3$  or  $4$  then the result is true so we assume  $|V| \geq 5$ . Consider a planar drawing of  $G$  for which  $a_1a_2a_3$  is the outer face. Then the neighborhood of  $a_1$  is a cycle  $a_3 = w_0w_1 \dots w_mw_{m+1} = a_2$ . Since  $G$  is a triangulation, there exists  $w_i$  with  $1 \leq i \leq m$  such that  $a_1$  and  $w_i$  have exactly two common neighbors, namely  $w_{i-1}$  and  $w_{i+1}$ . (If the cycle  $C_0 = w_0w_1 \dots w_mw_{m+1}$  is chordless we may choose  $w_i = w_1$ , otherwise we may choose  $w_i$  such that  $w_{i-1}w_j$  is a shortest chord of  $C_0$ , where  $j > i$ .) Then the neighbors of  $w_i$  in  $G$  form a cycle  $a_1z_0z_1 \dots z_dz_{d+1}$  where  $z_0 = w_{i-1}$  and  $z_{d+1} = w_{i+1}$  (see Figure 1(a)).

Let  $G'$  be the planar triangulation obtained from  $G$  by contracting the edge  $a_1w_i$  and giving the resulting vertex the label  $a_1$ . Then the neighborhood of  $a_1$  in  $G'$  is the cycle  $a_3 = w_0w_1 \dots w_{i-1}z_1 \dots z_dw_{i+1} \dots w_mw_{m+1} = a_2$  (see Figure 1(b)). Since  $a_1a_2a_3$  is a face of  $G'$ , by induction there exists a standard representation  $R' = R'(a_1, a_2, a_3)$  of  $V' = V \setminus \{w_i\}$  such that  $G' = \Sigma_2(R')$ .

We claim that  $w_0 <'_2 \dots <'_2 w_{i-1} <'_2 z_1 <'_2 \dots <'_2 z_d <'_2 w_{i+1} <'_2 \dots <'_2 w_{m+1}$  in  $R'$ . To see this, observe that by Lemma 2(2) and (4) applied to  $R'$ , there is a cycle  $C$  with vertex set  $\Gamma(a_1)$  in  $\Sigma_2(R')$ , that contains the edge  $a_2a_3$ , and the vertices in this cycle appear in increasing order in  $<'_2$  starting with  $w_0 = a_3$  and ending with  $w_{m+1} = a_2$ . Then since  $\Sigma_2(R')$  is a planar triangulation, by uniqueness of this cycle we must have  $C = w_0w_1 \dots w_{i-1}z_1 \dots z_dw_{i+1} \dots w_mw_{m+1}$ . This proves our claim.

By Lemma 2(1) applied to  $R'$  we find  $w_{m+1} <'_3 \dots <'_3 w_{i+1} <'_3 z_d <'_3 \dots <'_3 z_1 <'_3 w_{i-1} <'_3 \dots <'_3 w_0$ . We define linear orders  $R = \{<_1, <_2, <_3\}$  of  $V$  from  $R' = \{<'_1, <'_2, <'_3\}$  of  $V'$  as follows: we place  $w_i$  just below  $a_1$  in  $<'_1$  to form  $<_1$ . To form  $<_2$  we place  $w_i$  just above  $w_{i-1}$  in  $<'_2$ , and we form  $<_3$  by placing  $w_i$  just above  $w_{i+1}$  in  $<'_3$ .

Now to complete the proof we verify that  $G = \Sigma_2(R)$ . We observe that  $R = R(a_1, a_2, a_3)$  is standard by construction. Since  $R'$  represents  $V'$ , to check that  $R$  represents  $V$  we just need to verify that  $w_i$  occurs above every  $y$  in some order and below every  $y$  in some order. For  $y = a_1$  this is immediate as  $R$  is standard. For  $y \neq a_1$  we know that  $y <_1 w_i$ . Suppose on the contrary that  $y <_2 w_i$  and  $y <_3 w_i$ . Then by our construction  $y \leq_2 w_{i-1}$  and  $y \leq_3 w_{i+1}$ . Then  $y$  is not above both  $a_1$  and  $w_{i-1}$ , or above both  $a_1$  and  $w_{i+1}$ , in any order  $<'_1, <'_2, <'_3$ , contradicting the fact that  $a_1w_{i-1}$  and  $a_1w_{i+1}$  are edges of  $\Sigma_2(R')$ . Thus  $R$  represents  $V$ .

Let  $uv$  be an edge of  $G$ . If  $uv \in G' = \Sigma_2(R')$  then every element of  $V' = V \setminus \{w_i, u, v\}$  occurs above  $u$  and  $v$  in some order  $<'_k$  and therefore also in  $<_k$ . Moreover  $w_i$  occurs above  $u$  and  $v$  in  $<_1$  by construction, unless one of them is  $a_1$ , say  $u = a_1$ . Since  $a_1v \in G' = \Sigma_2(R')$  we know  $v = w_j$  for some  $j$ . If  $j \leq i-1$  then  $a_1, v <_2 w_i$  and if  $j \geq i+1$  then  $a_1, v <_3 w_i$ . Therefore  $uv$  is an edge of  $\Sigma_2(R)$ . If  $uv \notin G' = \Sigma_2(R')$  then by definition of contraction we have say  $u = w_i$ , and  $v \in \{a_1, z_0, \dots, z_{d+1}\}$ . Now  $a_1w_i \in \Sigma_2(R)$  because  $R$  represents  $V$ , and so  $w_i <_2 y$  or  $w_i <_3 y$  for every  $y \in V \setminus \{a_1, w_i\}$ . To see that  $w_iz_j \in \Sigma_2(R)$ , observe that since  $a_1z_j \in \Sigma_2(R')$  by definition of contraction, every element  $y$  of  $V \setminus \{w_i, z_j, a_1\}$  occurs above  $a_1$  and  $z_j$  in some order  $<'_k$ . Then  $k \in \{2, 3\}$ , and so also  $w_i, z_j <_k y$  by our placement of  $w_i$ . Finally note that  $w_i, z_j <_1 a_1$ . Therefore  $w_iz_j \in \Sigma_2(R)$ .

Thus we have shown that every edge of  $G$  is in  $\Sigma_2(R)$ . By Theorem 4 we know that  $\Sigma_2(R)$  is a planar triangulation, and since  $G$  is also a planar triangulation we conclude  $G = \Sigma_2(R)$ . This completes our proof.  $\square$

We end with the remark that there are other properties of planar triangulations  $G$  that are nicely captured by properties of a standard representation  $R$  of  $V = V(G)$ . For example, analogously to the graph  $\Sigma_2(R)$  one can define a 3-uniform hypergraph

$\Sigma_3(R)$  with vertex set  $V$  by letting  $xyw$  be an edge of  $\Sigma_3(R)$  if and only if for every  $z \in V$  there exists  $i \in \{1, 2, 3\}$  such that  $z \geq_i x$ ,  $z \geq_i y$  and  $z \geq_i w$ . It is quite easy to show that if  $R = R(a_1, a_2, a_3)$  is a standard representation of a vertex set  $V$ , and  $G$  is an embedding of  $\Sigma_2(R)$  in which  $a_1a_2a_3$  is the outer face, then  $\Sigma_3(R) = F(G) \setminus \{a_1a_2a_3\}$ , where  $F(G)$  denotes the set of faces of  $G$ .

## Acknowledgments

The authors are grateful to Tom Trotter for helpful discussions.

## References

- [1] L. Babai, D. Duffus, Dimension and automorphism groups of lattices, *Algebra Universalis* 12 (1981), 279–289.
- [2] G. Brightwell, W.T. Trotter, The order dimension of convex polytopes, *SIAM J. Discrete Mathematics* 6 (1993), 230–245.
- [3] G. Brightwell, W.T. Trotter, The order dimension of planar maps, *SIAM J. Discrete Mathematics* 6 (1997), 515–528.
- [4] L. Castelli Aleardi, E. Fusy, T. Lewiner, Schnyder woods for higher genus triangulated surfaces, with applications to encoding. *Discrete Comput. Geom.* 42 (2009), 489–516.
- [5] S. Felsner, Convex drawings of planar graphs and the order dimension of 3-polytopes, *Order* 18 (2001), 19–37.
- [6] S. Felsner, Geodesic embeddings and planar graphs, *Order* 20 (2003), 135–150.
- [7] S. Felsner, W.T. Trotter, Posets and planar graphs, *J. Graph Theory* 49 (2005), 273–284.
- [8] S. Felsner, F. Zickfeld, Schnyder woods and orthogonal surfaces. *Discrete Comput. Geom.* 40 (2008), 103–126.
- [9] H. de Fraysseix, J. Pach, R. Pollack, How to draw a planar graph on a grid, *Combinatorica* 10 (1990), 41–51.
- [10] P. Ossona de Mendez, Geometric realization of simplicial complexes, *Graph Drawing, Lecture Notes in Computer Science* 5 (1999), 323–332.
- [11] W. Schnyder, Planar graphs and poset dimension, *Order* 5 (1989), 323–343.